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Surface waves in flowing water

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Surface waves in flowing water and their stability are studied. With $U(y)$ denoting the mean velocity and d the depth of water, the following results are obtained: (i) in the plane of the complex wave velocity, $c = c_r + ic_i$, all eigenvalues c with a positive c_i lie within a semicircle which has as its diameter the range of the velocity $U(y)$ of the primary flow, y being the vertical co-ordinate. (ii) If $U''(y)$ does not change sign and U is monotonic in the field of flow, singular neutral modes (for which $c = U$ somewhere in the field of flow) are impossible and the flow is stable. (iii) If U is analytic and U'' vanishes at the point or points where U is equal to the same constant U_c and where U' is not zero then at least one neutral mode exists with $c = U_c$, provided $U(d) \neq U_c$. (iv) If U is monotonic and $U''/(U-c)$ is finite and non-zero at the critical point (c real), where U'' vanishes, then the neutral mode mentioned in (iii) above is contiguous with unstable modes. (v) If $U'' < 0$ and $U' \geqslant 0$ there are waves with $c \leqslant U(0)$, with a finite maximum wavenumber k_c corresponding to $c = U(0)$ and with c decreasing monotonically to a finite c_0 for $k = 0$. (vi) If $U'' < 0$ and $U' \geqslant 0$ waves of all wavenumbers can travel with $c > U(d)$. The eigenvalue c for any k is bounded.

1. Introduction

The question of surface waves in flowing water is inseparable from the question of their stability. In previous studies of this question, by Burns (1953), Hunt (1955) and Benjamin (1962), the question of stability has not been posed. With $U(y)$ denoting the velocity of the primary flow at elevation y, Burns found that if $U''(y)$ is non-positive and $U(y)$ increases monotonically with y then two real values of the wave velocity exist for *long* waves, one of which is greater than the surface velocity $U(d)$, d being the depth of water, and the other less than $U(0)$. Benjamin (1962) questioned the existence of the smaller $c < U(0)$ claimed by Burns. In his criticism Benjamin seems to believe that for $U''(y) < 0$ a singular neutral mode with $U(0) < c < U(d)$ exists when $U(d)$ is very much greater than the speed of very long waves in quiet water. However, Burns's conclusion is supported by the present work. Hunt was not concerned with whether a neutral mode may be singular in his calculation of (real) eigenvalues of c for a seventh-power law for U ($U = Cy^{\frac{1}{7}}$). It will be seen in this paper that unless $U''(y)$ vanishes somewhere in the field of flow there cannot be singular neutral modes with $c = U(y)$ for $0 < y < d$.

The present work is concerned not only with surface waves but their stability. Where it deals with neutral waves the approach is different from those of previous writers.

2. Governing equations

If p denotes the pressure perturbation, that is, the *deviation* of the pressure from the hydrostatic pressure, ρ denotes the (constant) density, t denotes the time, and x and y are Cartesian co-ordinates, with y measured in the vertical direction, then the linearized equations of motion of a liquid, with viscous effects neglected, are

$$\rho(u_t + Uu_x + U'v) = -p_x, \tag{1}$$

$$\rho(v_t + Uv_x) = -p_y, \tag{2}$$

where u and v are the components of the velocity perturbation in the directions of increasing x and y, respectively, U is the mean velocity, which is in the direction of x and is a function of y only, subscripts denote partial differentiation and primes on U denote differentiation with respect to y. The equation of continuity

$$u_x + v_y = 0$$

allows one to use the stream function ψ in terms of which

$$u = \psi_y, \quad v = -\psi_x. \tag{3}$$

We shall assume

$$\psi = f(y)\, e^{ik(x-ct)}, \tag{4}$$

where

$$c = c_r + ic_i \tag{5}$$

is the wave velocity. Eliminating p between (1) and (2) and using (3) and (4) we arrive at the well-known equation

$$f'' + [U''/(c-U) - k^2]f = 0. \tag{6}$$

It is sometimes more convenient to use the function $F(y)$ defined by

$$f(y) = (c-U)\,F(y).$$

In terms of $F(y)$, (6) becomes, as can be readily verified,

$$[(U-c)^2 F']' - k^2(U-c)^2 F = 0. \tag{7}$$

The boundary condition for f at the bottom, where $y = 0$, is

$$f(0) = 0. \tag{8}$$

At the free surface the condition is

$$p(d) - \rho g\eta = 0, \tag{9}$$

where η is the displacement of the free surface from its mean position. Since the kinematic condition at the free surface is

$$\eta_t + U\eta_x = v = -\psi_x, \tag{10}$$

from (1), (9) and (10) we obtain, remembering that p also has the factor $\exp ik(x-ct)$,

$$f'(d) = \left[\frac{g}{(U-c)^2}+\frac{U'}{U-c}\right]f(d), \tag{11}$$

In terms of $F(y)$, (8) and (11) become respectively

$$F(0) = 0 \tag{12}$$

and

$$(U-c)^2F'(d) = gF(d). \tag{13}$$

3. The semicircle theorem

By a slight extension Howard's semicircle theorm (1961) can be generalized to apply to flows with a free surface. With $W = U-c$ we can write (7) as

$$(W^2F')'-k^2W^2F = 0. \tag{14}$$

Multiplying this equation by F^*, integrating from zero to d and using (12) and (13) whenever necessary, we have

$$\int W^2(|F'|^2+k^2|F|^2)\,dy-g|F(d)|^2 = 0, \tag{15}$$

where the limits of integration are understood. The real and imaginary parts of (15) are

$$\int [(U-c_r)^2-c_i^2]\,Q\,dy-g|F(d)|^2 = 0 \tag{15a}$$

and

$$2c_i\int (U-c_r)\,Q\,dy = 0, \tag{15b}$$

where

$$Q = |F'|^2+k^2|F|^2. \tag{16}$$

If c_1 is not zero then

$$\int UQ\,dy = \int c_rQ\,dy \tag{17}$$

and (15a) can be written as

$$\int U^2Q\,dy = (c_r^2+c_i^2)\int Q\,dy+g|F(d)|^2. \tag{18}$$

If a and b are the minimum and maximum of U respectively, so that $a \leqslant U \leqslant b$, it is obvious that

$$0 \geqslant \int (U-a)(U-b)\,Q\,dy = \int U^2Q\,dy-(a+b)\int UQ\,dy+ab\int Q\,dy.$$

Using (17) and (18) we then have

$$0 \geqslant [c_r^2+c_i^2-(a+b)\,c_r+ab]\int Q\,dy+g|F(d)|^2,$$

from which it follows that

$$[c_r-\tfrac{1}{2}(a+b)]^2+c_i^2 \leqslant [\tfrac{1}{2}(a+b)]^2-ab = [\tfrac{1}{2}(b-a)]^2. \tag{19}$$

Hence the semicircle theorem:

THEOREM 1. The complex wave velocity c for any unstable mode must lie inside the semicircle in the upper half of the complex c plane which has the range of U for diameter.

4. The Reynolds stress

We have just seen that if c_r is outside of the range of U then c_i must be zero. Such wave modes, if they exist, are called non-singular modes. On the other hand, if c_r falls within the range of U then c_i may or may not be zero. Such modes, if $c_i = 0$, are called singular neutral modes. In order to study singular neutral modes and the unstable modes contiguous to them (as we shall see later) we consider the Reynolds stress in the fluid defined by

$$\tau = -\rho\overline{uv}, \tag{20}$$

the bar indicating a time average. Remembering that u and v are the real parts of $f'(y)\, e^{ik(x-ct)}$ and $-ikf(y)\, e^{ik(x-ct)}$ respectively we easily obtain

$$\tau = \tfrac{1}{2}\rho k \operatorname{Im}(f^*f')\, e^{2kc_it} = (\rho k/4i)(f^*f' - ff^{*\prime})\, e^{2kc_it}, \tag{21}$$

where f^* is the complex conjugate of f. If we differentiate (21) and use (6) we obtain the expression of Foote & Lin (1950):

$$\frac{d\tau}{dy} = \frac{1}{k}\rho\overline{v^2}\, c_i \frac{U''}{|U-c|^2}. \tag{22}$$

If now we let c_i approach zero we see that $d\tau/dy$ approaches zero everywhere except at the critical point, where $U = c_r$. Integration of (22) across the critical point yields the jump† in τ as c_i approaches zero:

$$[\tau] = \tau(y_c^+) - \tau(y_c^-) = (\pi/k)\,\overline{v^2}\, U_c''/|U_c'|, \tag{23}$$

which is given by Lin (1955, p. 54), the subscript c meaning 'critical'. Thus for a singular neutral mode τ is constant between critical points, but suffers a jump as given by (23) across a critical point.

Now consider the Reynolds stress at the free surface. From (11) and (21) it follows that at the free surface

$$\tau = \tfrac{1}{2}\rho k\, c_i \left[\frac{g(U-c_r)}{|U-c|^4} + \frac{U'}{|U-c|^2}\right] |f|^2, \tag{24}$$

where all variables depending on y are to be evaluated at $y = d$. It is evident that $\tau = 0$ if $c_i = 0$. Hence for a neutral mode the Reynolds stress is zero at the free surface. At the bottom, where f is zero, the Reynolds stress is zero for any mode, neutral or unstable.

The jump in τ at the critical point is zero if U_c' is not zero but either U_c'' or v is zero there. If U is monotonic there can be only one critical point. If at that point

† One referee of this paper thinks the validity of (23) depends on the existence of unstable modes with complex values of c in the neighbourhood of the real c under discussion. The author of this paper does not share this view but chooses to record the disagreement here.

U'' is not zero and v cannot be zero then singular neutral mode for that c (real) is not possible, since there is a jump in τ which is, however, prohibited by the boundary conditions.

5. Sufficient conditions for stability

Multiplying (6) by f^* and integrating in the fluid domain, we have, upon using (8) and (11),

$$-\int (|f'|^2+k^2|f|^2)\,dy-\int \frac{U''}{U-c}|f|^2dy+\left(\frac{g}{(U-c)^2}+\frac{U'}{U-c}\right)_d|f(d)|^2=0, \quad (25)$$

in which the subscript d indicates that the bracket is to be evaluated at $y=d$. After multiplication by -1, the imaginary part of (25) is

$$c_i\left\{\int \frac{U''}{|U-c|^2}|f|^2-\left(\frac{2g(U-c_r)}{|U-c|^4}+\frac{U'}{|U-c|^2}\right)_d|f(d)|^2\right\}=0. \quad (26)$$

If $$U''<0 \quad \text{and} \quad U'(d)\geqslant 0, \quad (27)$$

and we suppose $c_i \neq 0$, then (26) demands

$$U(d)=\max U<c_r. \quad (28)$$

But, by theorem 1, if (28) is satisfied c_i must vanish, leading to a contradiction of the supposition $c_i \neq 0$. Hence c_i must be zero if (27) is satisfied and we have

THEOREM 2 (*a*). If $U''<0$ throughout the fluid domain and $U'(d)\geqslant 0$, the free-surface flow is stable.

By almost identical arguments we also have

THEOREM 2 (*b*). If $U''>0$ throughout the fluid domain and $U'(d)\leqslant 0$, the free-surface flow is stable.

Actually theorem 2 (*b*) is an obvious consequence of theorem 2 (*a*) since upon reversing the positive direction of flow theorem 2 (*b*) becomes theorem 2 (*a*).

6. Singular neutral modes

If (27) is satisfied we can show that singular neutral modes are impossible. Let c be real and $U=c$ at $y=y_c>0$. First, we note that $v\neq 0$ (i.e. $f\neq 0$) at this point. For if $f(y_c)$ were zero, then multiplying (6) by f^* and integrating between zero and y_c we would have

$$\int_0^{y_c}(|f'|^2+k^2|f|^2)\,dy+\int_0^{y_c}\frac{U''}{U-c}|f|^2dy=0, \quad (29)$$

which is evidently absurd since $U\leqslant c$ and $U''<0$ in the domain of integration. Hence $f(y_c)\neq 0$. Note that if $f(y_c)=0$, f contains only the solution f_1 which is analytic at y_c and contains the factor $(y-y_c)$. Hence all the integrals are convergent. The other solution of (6) is of the form

$$f_2=1+\ldots+\frac{U_c''}{U_c'}f_1(y-y_c)\ln(y-y_c),$$

as is well known. Then if (27) is satisfied the jump in τ across the critical point (if any) given by (23) cannot be zero. Since τ is zero both at $y = 0$ and at $y = d$, this jump cannot happen. Hence there cannot be a critical point, i.e. there cannot be a singular neutral mode if (27) is satisfied. The same is true if $U'' > 0$ and $U' \leqslant 0$. Hence we have

THEOREM 3. If U is monotonic and the U–y curve has no point of inflexion, singular neutral modes are impossible.

Note that since $f(0) = 0$, if $y_c = 0$ the solution must be the non-singular one f_1.

We shall now show that a singular neutral mode exists if U is analytic and monotonically increasing and $U''(y)$ vanishes at some point in the fluid domain. In fact, the c will be the U at the point where U'' vanishes. For then

$$K(y) = U''/(c-U) \tag{30}$$

is analytic everywhere, including the critical point y_c. We can then apply the Sturm–Liouville theory to (6) and its boundary conditions. With

$$U(y_c) = c \quad \text{and} \quad U''(y_c) = 0, \tag{31}$$

a solution satisfying (8) can always be found for any k. But we have to show that (11) can be satisfied for that c and some real value of k, with (8) satisfied. To do so we multiply (6) by $U-c$ and integrate from zero to d, obtaining

$$[U(d)-c]f'(d)-[U(0)-c]f'(0)-f(d)\,U'(d)-k^2\int(U-c)f\,dy = 0. \tag{32}$$

We may assign the value 1 to $f'(0)$, since the eigenfunction can be multiplied by an arbitrary constant, and for convenience denote the positive number $c-U(0)$ by the symbol c'. Then (32) can be written as

$$\frac{f'}{f} = \frac{U'}{U-c} - \frac{c'}{(U-c)f} + \frac{k^2}{(U-c)f}\int_0^d (U-c)f\,dy, \tag{33}$$

in which all functions of y, except those inside the integration sign, are evaluated at $y = d$, where, one recalls, $U-c$ is positive. We know from the Sturm theory that if k^2 increases $f'(d)/f(d)$ will increase. For $k = 0$,

$$\frac{f'}{f} = \frac{U'}{U-c} - \frac{c'}{(U-c)f}, \tag{33a}$$

with all functions of y evaluated at d. Very near y_c, just above it, $U'/(U-c)$ is positive and as large as we please, and hence is greater than† f'/f evaluated at

† This is so because $f(y_c)$ cannot be zero, as can be seen from an equation similar to (32), with y_c replacing d.

$$[U(y_c)-c]f'(y_c)-[U(0)-c]f'(0)-f(y_c)\,U'(y_c)-k^2\int_0^{y_c}(U-c)f\,dy = 0.$$

The dominant term in $f'(y)$ near y_c comes from $f_2'(y)$ and is equal or proportional to

$$U_c''\ln(y-y_c)/U_c',$$

since the expansion of $f_1(y)$ near y_c starts with the term $(y-y_c)$. The term $U(y)-c$ behaves like $U'(y_c)\,(y-y_c)$ near y_c since $U(y)$ is analytic. Hence the first term in the equation vanishes. If we set k equal to zero then $f(y_c)$ cannot be zero, since the second term in the equation is not zero. Hence f'/f can only be logarithmically large, whereas $U'/(U-c)$ can be large like $(y-y_c)^{-1}$.

the same place. Sturm's second comparison theorem then says that at $y = d$ $U'/(U-c)$ must be greater than f'/f, since for $k = 0$ we have

$$f'' - \frac{U''}{U-c} f = 0 \tag{34}$$

and

$$(U-c)'' - \frac{U''}{U-c}(U-c) = 0, \tag{35}$$

so that f and $U-c$ satisfy the same differential equation. According to this conclusion, since $c'/(c-U)$ is negative $f(d)$ must be positive as can be seen from (33a). In fact $f(y)$ cannot vanish for $y > y_c$, for otherwise we would have a $f'(y)/f(y)$ as large as we pleased above or below the zero of $f(y)$. Hence $f(d) \neq 0$ for $k = 0$, even if $f(y)$ is not the eigenfunction. If $f(y)$ is the eigenfunction then the free-surface condition forbids $f(d)$ to vanish. (From Sturm's first comparison theorem we also see that f cannot vanish below y_c except once, at $y = 0$. Thus for $k = 0$, $f(y)$ vanishes only at $y = 0$.) Hence for a very small k the two last terms in (33) definitely have a negative sum. The question then is whether $f'(d)/f(d)$ will increase to the value specified by (11) for the c under consideration as k^2 increases from zero, that is, whether the sum of the last two terms in (33) will reach $g/(U-c)^2$ evaluated at d. To answer this question, let us see how those terms behave at large k^2. Recalling that the $K(y)$ given by (30) is analytic let us denote its maximum value by M and its minimum value by m. Since f is made to vanish at $y = 0$, and $f'(0) = 1$, it can be readily shown that for $k^2 > M$,

$$\sinh(k''y)/k'' \leqslant f(y) \leqslant \sinh(k'y)/k', \tag{36}$$

where $k'' = (k^2 - M)^{\frac{1}{2}}$ and $k' = (k^2 - m)^{\frac{1}{2}}$.

In fact (36) is a consequence of Sturm's second comparison theorem, stating that

$$k'' \coth k''y \leqslant f'(y)/f(y) \leqslant k' \coth k'y. \tag{37}$$

Integration of (37) gives (36). The integral in (33), denoted by I, can be written as

$$I = I_1 + I_2,$$

where

$$I_1 = \int_0^{y_c} (U-c) f\, dy, \quad I_2 = \int_{y_c}^{d} (U-c) f\, dy. \tag{38}$$

Since f is positive for $k^2 > M$, on inspection of (6), with $f'(0) = 1$, we see that I_1 is negative and I_2 positive for $k^2 > M$. Let us define two numbers α and β such that

$$\alpha(y_c - y) > c - U \quad \text{for} \quad y \leqslant y_c, \tag{39}$$

$$\beta(y - y_c) < U - c \quad \text{for} \quad y_c \leqslant y. \tag{40}$$

Then

$$I > \int_0^{y_c} \alpha(y-y_c)\frac{\sinh k'y}{k'} + \int_{y_c}^{d} \beta(y-y_c)\frac{\sinh k''y}{k''}$$

$$= \frac{\beta}{k''^2}(d-y_c)\cosh k''d - \frac{\beta}{k''^3}(\sinh k''d - \sinh k''y_c) - \frac{\alpha y_c}{k'^2} - \frac{\alpha}{k'^3}\sinh k'y_c. \tag{41}$$

For large k^2, then, the integral I behaves like

$$\frac{\beta(d-y_c)}{k^2}\cosh kd,$$

provided $d \neq y_c$, and the last term in (33) behaves like

$$\frac{k\beta}{U(d)-c}(d-y_c)\coth kd. \tag{42}$$

The term in (33) containing c' behaves like

$$-\frac{c'}{U(d)-c}\frac{k}{\sinh kd}$$

for large k and is negligible compared with (42). As k increases, (42) increases without bound. Hence, being deficient at $k = 0$, as compared with (11), $f'(d)/f(d)$ must finally reach the value specified in (11) as k increases. Hence a singular neutral mode exists under the conditions stated and with c given by (31).

We have, for convenience of exposition, assumed U to be monotonically increasing. The same conclusion is reached if it is monotonically decreasing. In fact, the conclusion still holds even if U is not monotonic, provided that U'' vanishes at all points where $U = c$. We shall sketch the essence of the proof as follows.

(i) If there is more than one point at which $U = c$, $U-c$ vanishes more than once in $0 \leqslant y < d$. Hence using (34) and (35) for comparison we know that for $k = 0$ f vanishes at least once in the same interval (not including $y = 0$, where f is always zero). Hence on increasing k we can always make $f(d) = 0$ and $f'(d) < 0$. On increasing k a little more, it is evident that $f'(d)/f(d)$ can be made as near minus infinity as we please. Therefore the 'deficiency' of $f'(d)/f(d)$ as compared with its value demanded by (11) is established for some $k^2 = k_0^2 > 0$.

(ii) As k^2 increases beyond k_0^2, f is always positive for $y > 0$. Then, by a procedure similar to the one we have used above to discover the behaviour of I for large k^2 we can *always* show that the last term in (33) increases without bound as k^2 increases indefinitely, provided $U(d) \neq c$, whether $U(d)-c$ is positive or negative. It can in fact be shown that the integral I is dominated by that part of it which is between the largest y_c and d, and that the last term in (33) is positive and increases without bound as k^2 increases indefinitely. Hence the 'deficiency' mentioned in (i) can always be exactly eliminated, arriving at a neutral mode with c equal to U at one or more points of the flow. This mode really should not be called 'singular' any more.

Hence we have

THEOREM 4. If U is analytic and U'' vanishes at the point or points where U is equal to the same constant U_c and where U' does not vanish, then at least one neutral mode exists with $c = U_c$, provided $U(d) \neq U_c$.

The reason for using the words 'at least' is that if there is more than one point at which $U = U_c$ and $U'' = 0$, (11) may well be satisfied for more than one positive value of k^2. We have only shown that at least one such value exists. It can also be shown that for any k the eigenvalue c cannot be $U(d)$, whether $y = d$ is a critical point or not.

7. Unstable modes contiguous to singular neutral modes

If we vary k^2 slightly from its eigenvalue for the singular neutral modes mentioned in theorem 4 we obtain unstable modes. The demonstration is entirely similar to that of Lin (1955, pp. 122–123), the difference in the upper boundary condition causing no trouble whatever. We shall not reproduce Lin's analysis but shall only quote the results that can be obtained by his approach:

$$-dk^2/dc = A+iB, \tag{43}$$

where A and B are real and

$$B\int_0^d f_s^2\,dy = \pi\sum_i [K(y_c)f_s^2(y_c)]_i, \tag{44}$$

the summation being over all the critical points (where U'' vanishes and $U = c$). f_s is the eigenfunction for the 'singular' neutral mode. The free-surface boundary condition affects A but not B, except through f. If there is only one y_c and $K(y_c)$ is not zero, B is not zero and a slight change in k^2, be it positive or negative, will produce a positive c_i. Otherwise B may be zero, in which case a change in k^2 produces only a change in c_r. Hence we have

THEOREM 5. If U is monotonic and $K(y_c)$ is not zero the neutral mode mentioned in theorem 4 is contiguous to unstable modes.

8. Non-singular neutral modes

We shall restrict our attention to flows with the following properties:

$$U''(y) < 0 \quad \text{and} \quad U' \geqslant 0. \tag{45}$$

There are two classes of waves: class 1, with $c \leqslant U(0)$ and class 2, with $c > U(d)$.

Class 1

Consider first the limiting case of $c = U(0)$. For this case the c' in (33) is zero, and comparison of (33) with (11) produces

$$\frac{g}{U(d)-U(0)} = \frac{k^2}{f(d)}\int_0^d [U(d)-U(0)]f(y)\,dy, \tag{46}$$

for $y = d$. Since f cannot vanish† in the domain of flow it is positive throughout, except at $y = 0$. The discussion in §6 shows that the right-hand side of (46) increases without bound with k. Hence for some $k = k_c$ (depending on the flow), (46) and hence (11) are satisfied.

For $k < k_c$ we have $c < U(0)$ and c decreasing algebraically as k decreases. This can be seen by using Sturm's second comparison theorem on (6). For if k decreases and c does not decrease, the coefficient of f in (6) increases and hence $f'(d)/f(d)$ decreases according to Sturm's second comparison theorem. On the other hand if c does not decrease [and $c < U(0)$] the same quantity, $f'(d)/f(d)$, does

† Compare the f in (*b*) with the $U-c$ in (35). Both vanish at $y = 0$ if $c = U(0)$ and $U-c$ has no other zero. Since $k^2 \geqslant 0$, f cannot have a second zero.

not decrease, according to (11), leading to a contradiction. Hence c decreases as k decreases. It is also easy to see that c is bounded for $k = 0$. For otherwise, from the differential equation and $f(0) = 0$, we have $f(y) = y$, so that $f'(d)/f(d) = 1/d$, whereas (11) gives $f'(d)/f(d) = 0$. Hence

THEOREM 6. Under conditions (45) there are waves with $c \leqslant U(0)$, with a finite maximum wavenumber $k = k_c$ corresponding to $c = U(0)$, and with c decreasing monotonically to a finite value c_0 for $k = 0$.

Class 2

We now consider wave propagating downstream for which $c > U(d)$. Since the expression in parenthesis in (6) is positive, f is non-oscillatory and is in fact monotonically increasing from zero, as y increases. Therefore it is easy to see that $f(d) > d$. Since f is only zero at $y = 0$ and is positive elsewhere, and since c' is positive, the equation

$$c' + k^2 \int_0^d (c-U) f\,dy = \frac{g}{c-U(d)} f(d),$$

obtained by comparison of (33) with (11), can always be satisfied, whatever the value of k. Again it can be shown that the c corresponding to any k is bounded. Hence we have

THEOREM 7. Under conditions (45) waves of all wavenumbers can travel downstream with $c > U(d)$. The eigenvalue c for any k is bounded.

The present results support those of Burns (1953) which are for long waves only. Benjamin doubted the existence of waves with $c < U(0)$ when the surface velocity is a long way supercritical and brought the amplitude of the waves and the possibility of separation into his arguments against the existence of such waves. I do not follow Benjamin's arguments. However his statement that "in fact the wave will be convected downstream at an absolute velocity not much different from $\bar{U} - C_0$" ($\bar{U}$ = mean U, C_0 = wave speed in quiet water) is contradicted by our theorem 3, since Benjamin's statements apply explicitly to $U'' < 0$.

For an intuitive understanding of the propagation of long waves against the stream it is helpful to consider the density of the kinetic energy of very long waves, $[f'(y)]^2$, with the factor $\frac{1}{2}\rho$ omitted. If U'' is negative throughout and U increases monotonically with y, then for $k = 0$ and $c < U(0)$ integration of (6) subject to the conditions $f(0) = 0$, $f'(0) = 1$ shows that $f'(y)$ decreases with y, initially at least. Hence the kinetic energy at the bottom is at least a relative maximum and one is not surprised that a long wave can propagate its energy against the current near the bottom, where the situation is advantageous. This is, of course, merely an attempt to understand what goes on in an intuitive imprecise way. Its imprecision should not be allowed to cast doubt on the theory, which is entirely independent of such an intuitive argument.

In an open channel the waves with $c < U(0)$ can only be observed if a wave maker oscillates in the flowing water without blocking the flow. A stationary obstacle placed in the stream cannot be expected to produce upstream propagating waves, since even for a subcritical stream with uniform U, for which upstream propagating waves are known to exist, no such waves can be seen upstream from the obstacle. This phenomenon is related to the fact that the phase velocity is greater than the group velocity and is an altogether different matter. It should not be used as a basis for doubting the validity of the present results.

We note that for the special case of simple shear $U'' = 0$, and the differential equation (6) becomes identical to that for potential flow. The flow is always stable and no mode is singular, but it is now possible to have any c between $U(0)$ and $U(d)$. The only difference from wave motion in quiet water arises through (11).

We note also that if viscosity is taken into account, waves propagating against the stream may well be damped out and unstable modes with c_r in the range of U, permitted by the present theory, may be damped by viscosity to become neutral. These possibilities were pointed out by Velthuizen & Wijngaarden (1969), who considered a horizontal flow of a viscous fluid with a free surface. Strictly speaking, such a flow cannot be maintained, since there is no energy source, and not only waves, but the flow itself, must in time be damped out. Their calculation, however, is not without significance over a (relatively) short time. With this in mind we recall the results of Benjamin (1957) and Yih (1963), who found that for a viscous liquid layer flowing down an inclined plane: (*a*) No long waves can propagate upstream. (*b*) Undamped long waves propagate downstream with a speed equal to twice the surface (maximum) velocity of the mean flow, giving no indication of the existence of a singular neutral mode in the inviscid limit.

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Appendix

A referee has pointed out an interesting physical explanation for the upstream propagation of long waves even if the surface velocity is high. In what follows the original idea and the conclusion are his and the derivation is mine.

In a co-ordinate system moving with the waves the flow is steady. Choosing x and the stream function ψ in that system as independent variables, the total horizontal velocity component along a streamline in that system is

$$\hat{U}(\psi) + \hat{u}(\psi, x),$$

in which (the notation used in the paper being used for the other quantities)

$$\hat{U}(\psi) = U(y) - c, \quad \hat{u} = u + \eta U'(y) = f'(y)\, e^{ikx} + \eta U'. \tag{A 1}$$

We note that

$$v = \hat{U}\eta_x$$

which, in view of (3) and (4), means that

$$\eta = -\frac{f(y)}{U-c}\, e^{ikx}, \tag{A 2}$$

that is to say that the $F(y)$ in (7) is the amplitude of η. Then from (A 1) and (A 2) we have

$$\hat{U}\hat{u} = (f'\hat{U} - fU')\, e^{ikx}. \tag{A 3}$$

Now for long waves the governing equation is (34), integration of which gives

$$(U-c)f' - U'f = C.$$

Thus

$$\hat{U}\hat{u} = C e^{ikx},$$

which is independent of ψ.

Since $\hat{u}^2$ is proportional to the kinetic energy (along a streamline) of perturbation for long waves, this kinetic energy is a maximum where $|\hat{U}|$ is a minimum. For a monotonically increasing $U(y)$ (>0) and for $c < 0$ (propagation upstream), $\hat{U} = |\hat{U}|$ is a minimum at $y = 0$. Thus the kinetic energy is large at and near $y = 0$ if $U(0) = 0$, $c < 0$, and $|c| \ll 1$, and it is understandable why it can propagate against the weak current $U(y)$ near $y = 0$.

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